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# Digital Hilbert Transformation A Collocation Design Method

User Documentation

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# 1 Abstract

The Hilbert transformation or  $90^\circ$  phase shift of signals is used in a variety applications in signal processing and communications where the analytical signal is required.

This tutorial describes a MATLAB™ toolbox for the approximate design of digital Hilbert transformers, employing allpass filters. The collocation method developed for this task leads to low order filters with excellent filter characteristics in a large frequency band.

This tutorial gives a brief overview using the MATLAB toolbox.

# 2 Toolbox Overview

The Hilbert transformation  $\mathcal{H}$  or  $90^\circ$  phase shift of signals is used for generating the analytical signal

$$x^+(t) = x(t) + \mathcal{H}\{x(t)\} = x_I(t) + j x_Q(t)$$

where  $x_I$  is the inphase and  $x_Q$  the quadrature phase, respectively. The Hilbert transformation or  $90^\circ$  phase shift can only be realized approximately. For radio frequencies, distributed devices such as quadrature couplers or Schiffman phase shifters are state of the art. They operate however sufficiently only in a very small frequency band. At frequencies of medium range, several analog circuits based on lumped devices have been proposed. Such circuits are, e.g., parallel lowpass/highpass filters, analog allpass filters, polyphase filters (both passive and active filters) such as the Gingell phase shifter etc.

At low and moderate frequencies, digital filters are superior to analog ones, because they are of low cost, do not suffer from device mismatches, are easy to implement etc. This tutorial deals with a MATLAB toolbox for designing digital Hilbert transformers based on allpass filters with predefined phase functions. The technique for designing the allpass filters is a weighted collocation method.

We consider two allpass filters  $H_{1,2}(z)$  of degree  $N$  and  $N - 1$ , respectively, where

$$H_1(z) = \frac{\sum_{m=0}^N a_{N-m} z^{-m}}{\sum_{n=0}^N a_n z^{-n}} \quad H_2(z) = z^{N-1} \quad (1)$$

with the vector of filter coefficients  $a \in \mathbb{R}^N$ . The nominator coefficients are in reversed order to the denominator coefficients. The second filter is only a tapped delay line with phase response  $\varphi_2(\Omega) = \arg\{H_2\} = -(N - 1)\Omega$ . The first allpass filter  $H_1(z)$  shall fulfill the phase condition

$$\varphi_1(\Omega) = \varphi_2(\Omega) - \pi/2 = \arg\{H_1\} = -(N - 1)\Omega - \pi/2$$

This phase constraint can only be approximated, which is done here by a collocation method with weights.

We define the inner product with weight function  $w$

$$\langle f, g \rangle_w := \frac{1}{2\pi} \int_{-\pi}^{\pi} w f g^* d\Omega$$

where  $w(\Omega) > 0$   $[-\pi, \pi[$ . As test functions we employ Dirac delta distributions  $\varphi_l = \delta(\Omega - \Omega_l)$ ,  $l = 0, \dots, L \geq N$ ,  $\Omega_l \in [-\pi, \pi[$ . Employing the sift property of the delta distributions one obtains

$$\langle g, \varphi_l \rangle_w = w(\Omega_l) \sum_{n=0}^N a_{N-n} e^{-j n \Omega_l} - w(\Omega_l) e^{j \varphi(\Omega_l)} \sum_{n=0}^N a_n e^{-j n \Omega_l}$$

Collecting all equations in a system of linear homogeneous equations, we get

$$[\langle g, \varphi_0 \rangle_w, \dots, \langle g, \varphi_L \rangle_w]^T = 0$$

which leads to  $L + 1$  equations for the  $N + 1$  coefficients. Hence, for  $L > N$  the system of equations is over determined. We calculate the solution of least Euclidean ( $l_2$ ) norm. Choosing a suitable weighting function  $w$  enforces the accuracy of the approximation at specific frequencies.

It is well known that IIR filters are stably implemented as a cascade of 1<sup>st</sup> and 2<sup>nd</sup> order filter structures. We overtake below the notion from the design of Lattice Wave Digital Filters (LWDF). The allpass filter with transfer function  $H_1$  is realized in a cascade, namely

$$H_1(z) = \prod_{i=1}^M H^{(i)}(z) \quad (2)$$

where each block  $H^{(i)}(z)$  is a partial allpass filter of 1<sup>st</sup> and 2<sup>nd</sup> order, which are referred to as adaptors

$$\begin{aligned} H^{(i)}(z) &= \frac{c_{i1} + z^{-1}}{1 + c_{i1} z^{-1}} \\ H^{(i)}(z) &= \frac{c_{i2} + c_{i1} z^{-1} + z^{-2}}{1 + c_{i1} z^{-1} + c_{i2} z^{-2}} \end{aligned} \quad (3)$$

Therein, the index  $i$  represents the  $i$ -th. structure of the cascade with corresponding transfer function  $H^{(i)}(z)$ . Employing an elaborate hardware implementation structure for the adaptors, the number of multipliers can be reduced approximately by a factor of two compared to the direct form implementation of (1).

### 3 Use of the Toolbox

The toolbox is easy to use. Firstly, one has to specify the filter order  $N$  or filter degree  $N - 1$

```
% order of allpass transfer function system
N =10;
```

In the next step one defines the normalized frequency range of operation of the Hilbert transformation. Note that for  $\Omega = 0$  and the Nyquist frequency  $\Omega = \pi$  the requirement for the  $90^\circ$  phase shift cannot be achieved structurally. Therefore, one must define a lower bound  $\Omega_D$  and upper bound  $\Omega_S$

```
% design params
Omega_D = 0.1*pi; Omega_S = pi - Omega_D;
```

In the next step, one defines a vector of length  $L > N$  of collocation frequency points in the interval  $[0, \pi]$  and a corresponding weight vector  $w$ . The higher the weight at a specific collocation frequency  $\Omega_i$ , the better is the phase approximation. The default values used in this toolbox were obtained from an intensive analysis and work in most cases extraordinary well. Therefore, these weights shall be changed with care. The best approximation in the  $l_2$  norm for the weighted collocation method and the adaptor coefficients are then calculated automatically. The stability of the filter is also investigated.

## 4 Results

### 4.1 $N = 10$ order Hilbert transformation

We consider a Hilbert transformer with order  $N = 10$  or degree 9. The Hilbert transformer shall operate in the normalized frequency range  $\Omega = [0.1\pi, 0.9\pi]$ . The weights within the frequency range of interest are chosen to be 40 dB as depicted in Fig. 1. The Fig. 2(a) depicts the negated phase functions  $-\arg\{H(e^{j\Omega})\}$ . Within the predefined frequency range the two phases are well  $90^\circ$  out of phase, which can also be seen by the group delay variation of the first allpass filter  $H_1(z)$  shown in Fig. 2(b). The hardware realization requires only 5 multipliers, as can be seen from the calculated filter coefficients  $a$

```
ad_S1 =
1.0000    0   -0.4854    0   -0.1043    0   -0.0392    0
   -0.0162    0   -0.0054
```

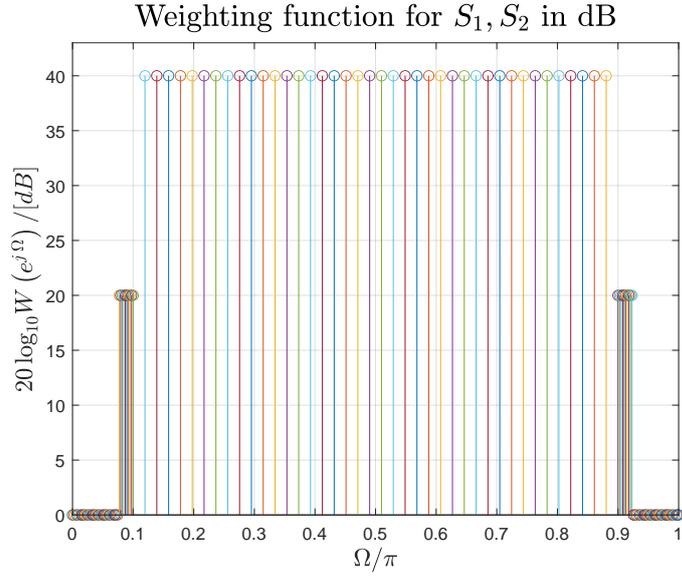
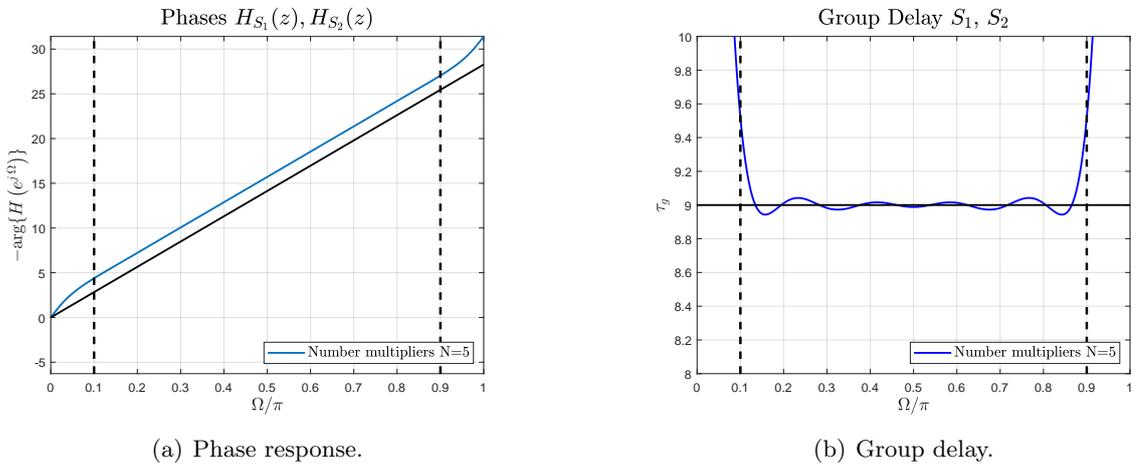


Figure 1: Weights of the collocation points in  $20 \log_{10} W (e^{j\Omega}) / [dB]$ .



(a) Phase response.

(b) Group delay.

Figure 2: Negated phases  $-\arg\{H(e^{j\Omega})\}$  and group delays of the allpass filters.